D = diffusivity of dissolved gas
d = diameter of packing particle
k_L = liquid-film coefficient for physical absorption
$\bar{k}_L$ = average value of k_L over entire surface of packing
L = liquid mass velocity
N = average rate of absorption per unit area of wetted surface in packing
$Q(t)$ = quantity of gas absorbed by unit area of stagnant liquid of effectively infinite depth in time t
r = first-order reaction-velocity constant
r' = second-order reaction-velocity constant
s = fractional rate of surface renewal
t = time of exposure of liquid to gas
α = ratio of N with reaction to N without reaction, other things equal
$\bar{\alpha}$ = average value of α over entire surface of packing
β = constant in Equation (20)
θ = effective exposure time of liquid on packing
μ = viscosity of liquid
ρ = density of liquid
$\phi(t)$ = distribution function for local surface ages
$\psi(k_L)$ = distribution function for local values of k_L

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Heat Transfer in Cylinders with Heat Generation

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The prediction of temperatures has been studied for a tubular flow reactor when heat is generated at a rate which is a linear function of the local temperature. Analytical solutions are presented both for the case where the wall is isothermal and for the case where the exterior surroundings are isothermal and the heat transfer coefficient between the tube wall and the surroundings is constant. This analysis should be helpful for estimating local temperatures and also for predicting the transient response to changes in one of the independent operating variables.

One of the fundamental chemical engineering problems is the prediction of temperatures in a flow reactor. The temperature pattern is dictated by the inlet temperature of the fluid, heat generation within the fluid due to chemical or nuclear reaction, heat transfer to the surroundings, the velocity distribution and transport properties of the fluid, and the geometry of the reactor. Every case of this problem has some unique aspects, but there are enough common features to make a simplified theoretical analysis desirable. The present analysis should be helpful for estimating temperatures in a flow reactor and for predicting the transient response to changes in an operating variable.

The assumption that the rate of heat generation depends only on the local temperature is valid for chemical reactions only when they are of zero order. The linear temperature dependence of the heat source may be viewed as an approximation of the exponential variation of chemical reaction rate with temperature.

The reactor is taken to be a tube of radius s, through which the fluid

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flows at uniform and steady velocity V. The fluid enters the tube at a uniform and steady temperature t_o. Heat is generated within the fluid at a rate which is a linear function of the local temperature. The tube is surrounded by a medium at the constant temperature t_s, and the heat transfer coefficient between the fluid and its environment is a constant, h. When h is infinite, it is implied that the fluid in contact with the tube wall is always at the constant temperature t_s.

The differential equation of heat convection with a volume heat source is (*2*)

$$\frac{dt}{d\tau} = \frac{\partial t}{\partial \tau} + V_x \frac{\partial t}{\partial x} + V_y \frac{\partial t}{\partial y} + V_z \frac{\partial t}{\partial z} = \frac{q}{\rho c} + \alpha \nabla^2 t \qquad (1)$$

Here t is temperature; τ is time; V_x, V_y, V_z are the components of the velocity in the x, y, z directions; q is the rate of heat generation per unit volume and time; ρ is the density; c is the heat capacity; and α is the thermal diffusivity of the fluid. The conditions assumed are

a. Steady state.

b. Symmetry of temperature about the axis.

c. Fluid velocity in x direction only (axial velocity) and equal to V.

d. Physical properties of the fluid independent of temperature.

e. Axial conductivity negligible ($\partial^2 t/\partial x^2 << \partial^2 t/\partial r^2 + 1/r \;\; \partial t/\partial r$)

$$\left(\frac{\partial^2 t}{\partial x^2} \ll \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right)$$

f. Heat generation a linear function of temperature: $q/\rho c = At + B$.

Then by use of cylindrical coordinates,

$$V\frac{\partial t}{\partial x} = \alpha\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right) + At + B \qquad (2)$$

The appropriate initial and boundary conditions for the differential equation (2) are (3) and either (4) or (5):

$$\text{At } x = 0, t = t_0 \qquad (3)$$

$$\text{At } r = s, t = t_s \text{ (isothermal wall)} \qquad (4)$$

$$\text{or } k\left(\frac{\partial t}{\partial r}\right)_{r=s} = -h\,(t-t_s)_{r=s} \qquad (5)$$

(heat loss to surroundings at constant temperature)

The solution to the problem of the constant source ($A = 0$) with boundary condition (4) has been presented (*3*) and is

$$\frac{t-t_s}{t_o-t_s} = \sum_{n=1}^{\infty} \frac{2}{\beta_n J_1(\beta_n)}\left[1 - \frac{Bs^2}{\alpha\,(t_o-t_s)\,\beta_n^2}\right] J_o(\beta_n w)\, e^{-\beta_n^2 \frac{\alpha}{sV}\frac{x}{s}} + \frac{Bs^2}{4\,\alpha\,(t_o-t_s)}(1-w^2) \qquad (6)$$

The J_0 and J_1 are Bessel functions of the first kind and of zero and first order respectively, and the β_n are the positive roots of the equation $J_0(\beta) = 0$. The first three β_n are 2.40 5.52 and 8.65.

When the boundary condition is (5) rather than (4) and the source term is constant ($A = 0$), the solution has the form

$$\frac{t-t_s}{t_o-t_s} = \sum_{n=1}^{\infty} N_n J_o(\lambda_n w) e^{-\lambda_n^2 \frac{\alpha}{sV}\frac{x}{s}} + \frac{1}{(t_o-t_s)}\left[\frac{B_s^2}{4\alpha}\left(1+2\frac{k}{sh}-w^2\right)-t_s\right] \qquad (7)$$

The λ_n are the positive roots of the equation

$$\lambda_n J_1(\lambda_n) - \frac{hs}{k} J_o(\lambda_n) = 0 \qquad (8)$$

These roots are tabulated as Table 3 of Carslaw and Jaeger (*1*). hs/k is a kind of Nusselt number, with h being the film heat transfer coefficient of the surroundings, and k the thermal conductivity of the fluid itself. The N_n are calculated to satisfy the initial condition (3). Use is made here of the Dini expansions of unity and of w^2 (*4*):

$$1 = 2\left(\frac{hs}{k}\right)\sum_{n=1}^{\infty} \frac{J_o(\lambda_n w)}{\left[\lambda_n^2 + \left(\frac{hs}{k}\right)^2\right] J_o(\lambda_n)} \qquad (9)$$

$$w^2 = 2\sum_{n=1}^{\infty} \frac{\left[2 + \frac{hs}{k} - 4\left(\frac{hs}{k}\right)\left(\frac{1}{\lambda_n}\right)^2\right] J_o(\lambda_n w)}{\left[\lambda_n^2 + \left(\frac{hs}{k}\right)^2\right] J_o(\lambda_n)} \qquad (10)$$

at $x = 0$

$$\frac{t-t_s}{t_o-t_s} = 1 = \sum_{n=1}^{\infty} N_n J_o(\lambda_n w) + \frac{1}{(t_o-t_s)}\left[\frac{Bs^2}{4\alpha}\left(1 + 2\frac{k}{sh} - w^2\right) - t_s\right] \qquad (11)$$

The Dini expansion on the right must equal

$$1 - \frac{Bs^2}{4\alpha\,(t_o-t_s)}\left(1+\frac{2k}{sh}-w^2\right) + \frac{t_s}{(t_o-t_s)}$$

so that

$$N_n = \frac{2}{\left[\lambda_n^2 + \left(\frac{hs}{k}\right)^2\right] J_o(\lambda_n)} \left\{\frac{hs}{k}\left[1 + \frac{t_s}{t_o-t_s} - \frac{Bs^2\left(1+\frac{2k}{sh}\right)}{4\alpha\,(t_o-t_s)}\right] + \frac{Bs^2\left[2+\frac{hs}{k} - 4\left(\frac{hs}{k}\right)\left(\frac{1}{\lambda_n}\right)^2\right]}{4\alpha\,(t_o-t_s)}\right\} \qquad (12)$$

and the solution is

$$\frac{t-t_s}{t_o-t_s} = \frac{1}{t_o-t_s}\left[\frac{Bs^2\left(1+\frac{2k}{sh}-w^2\right)-t_s}{4\alpha}\right] + 2\sum_{n=1}^{\infty}\frac{hs}{k\left[\lambda_n^2+\left(\frac{hs}{k}\right)^2\right]J_o(\lambda_n)}$$

$$\left\{\left[1+\frac{t_s}{t_o-t_s}-\frac{Bs^2\left(1+\frac{2k}{sh}\right)}{4\alpha\,(t_o-t_s)}\right]+\frac{Bs^2}{4\alpha\,(t_o-t_s)}\left[\frac{2k}{sh}+1-\frac{4}{\lambda_n^2}\right]\right\}$$

$$J_o(\lambda_n w)e^{-\lambda_n^2\frac{\alpha}{sv}\frac{x}{s}} \qquad (13)$$

Table 1 contains the first three roots of Equation (8) for some values of hs/k.

Figure 1 is a graphical representation of Equations (6) and (13) respectively in terms of various groups, $t_s = 0$ being assumed.

The solutions to the complete Equation (2) are somewhat more complex. When the wall is isothermal [boundary condition is Equation (4)], the solution has the form

$$t+\frac{B}{A} = \left(t_s+\frac{B}{A}\right)\frac{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]} + \sum_{n=1}^{\infty}\left[\frac{2\left(t_o+\frac{B}{A}\right)}{\beta_n J_1(\beta_n)}\right.$$

$$\left. +N_n\right] J_o(\beta_n w)e^{-\frac{x}{v}\left[A-\alpha\left(\frac{\beta_n}{s}\right)^2\right]} \qquad (14)$$

Equation (14) satisfies the boundary condition (4); it will satisfy the initial condition (3) if

$$\sum_{n=1}^{\infty} N_n J_o(\beta_n w) = -\left(t_s+\frac{B}{A}\right)\frac{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]} \qquad (15)$$

Use is made here of the Fourier-Bessel expansion of $J_o[(As^2/\alpha)^{\frac{1}{2}}w]$ (4):

$$J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right] = \sum_{n=1}^{\infty}\frac{2\,\beta_n J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]J_o(\beta_n w)}{\left(\beta_n^2-\frac{As^2}{\alpha}\right)J_1(\beta_n)} \qquad (16)$$

Then the solution to the problem of linearly temperature-dependent heat sources in a fluid whose wall is maintained at a constant temperature is

$$\frac{t-t_s}{t_o-t} = \frac{t_s+\frac{B}{A}}{t_o-t_s}\left\{\frac{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]}-1\right\} + \frac{2}{t_o-t_s}\sum_{n=1}^{\infty}\frac{1}{\beta_n J_1(\beta_n)}\left[\left(t_o\right.\right.$$

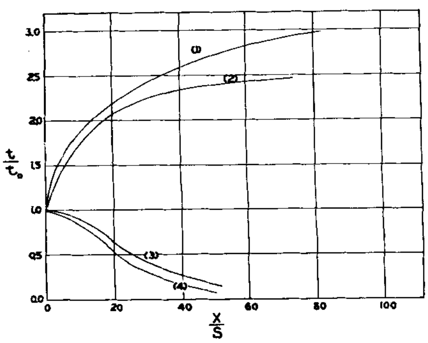

Fig. 1. Temperature at axis of tube. ($A = 0$, $t_s = 0$, $\alpha/sV = 10^{-2}$).

$$\frac{Bs^2}{\alpha t_o} = 10,\ \frac{hs}{k} = 10 \qquad (1)$$

$$\frac{Bs^2}{\alpha t_o} = 10,\ \frac{hs}{k} = \infty \qquad (2)$$

$$\frac{Bs^2}{\alpha t_o} = 0,\ \frac{hs}{k} = 10 \qquad (3)$$

$$\frac{Bs^2}{\alpha t_o} = 0,\ \frac{hs}{k} = \infty \qquad (4)$$

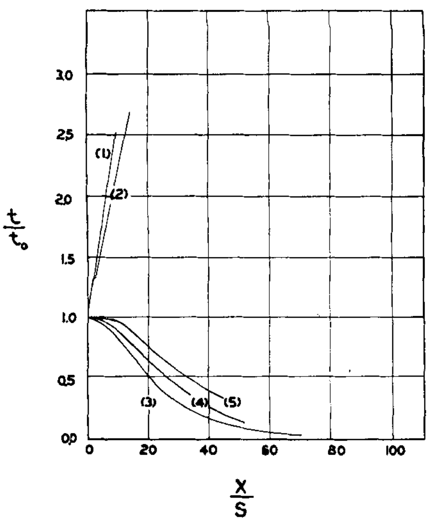

Fig. 2. Temperature at axis of tube, temperature-dependent volume heat source ($t_s = 0$, $B/A = 0$, $\alpha/sV = 10^{-2}$).

$$\frac{As}{V} = 10^{-1},\ \frac{hs}{k} = 10 \qquad (1)$$

$$\frac{As}{V} = 10^{-1},\ \frac{hs}{k} = \infty \qquad (2)$$

$$\frac{As}{V} = 0,\ \frac{hs}{k} = \infty \qquad (3)$$

$$\frac{As}{V} = 10^{-2},\ \frac{hs}{k} = \infty \qquad (4)$$

$$\frac{As}{V} = 10^{-2},\ \frac{hs}{k} = 10 \qquad (5)$$

TABLE 1.—ROOTS *of* EQUATION (8)

hs/k	λ_1	λ_2	λ_3
0.01	0.141	3.83	7.02
0.1	0.442	3.86	7.03
1.0	1.26	4.08	7.16
10.0	2.18	5.03	7.6
∞	2.40	5.52	8.65

$$+\frac{B}{A}\Bigg)-\frac{\left(t_s+\frac{B}{A}\right)\beta_n^2}{\left(\beta_n^2-\frac{As^2}{\alpha}\right)}\Bigg]J_o(\beta_n w)\,e^{\frac{x}{v}\left[A-\alpha\left(\frac{\beta_n}{s}\right)^2\right]} \tag{17}$$

With boundary condition (5) instead of (4), the solution to differential equation (2) is of the type

$$t+\frac{B}{A}=\frac{\left(t_s+\frac{B}{A}\right)J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{\left\{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]-\frac{k}{sh}\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}J_1\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]\right\}}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{2\left(t_o+\frac{B}{A}\right)\left(\frac{hs}{k}\right)}{\left[\lambda_n^2+\left(\frac{hs}{k}\right)^2\right]J_o(\lambda_n)}+N_n\right\}J_o(\lambda_n w)\,e^{\frac{x}{v}\left[A-\alpha\left(\frac{\lambda_n}{s}\right)^2\right]} \tag{18}$$

The N_n of (18) will satisfy the initial condition if

$$-\sum_{n=1}^{\infty}N_n J_o(\lambda_n w)=\frac{\left(t_s+\frac{B}{A}\right)J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{\left\{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]-\frac{k}{sh}\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}J_1\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]\right\}} \tag{19}$$

Use is made of the Dini expansion of

$$J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]:$$

$$J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]=$$

$$2\left(\frac{hs}{k}\right)\sum_{n=1}^{\infty}\frac{\lambda_n^2\left\{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]-\frac{k}{sh}\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}J_1\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]\right\}J_o(\lambda_n w)}{\left[\lambda_n^2+\left(\frac{hs}{k}\right)^2\right]\left(\lambda_n^2-\frac{As^2}{\alpha}\right)J_o(\lambda_n)} \tag{20}$$

When (20) is used with (19), (18) becomes

$$\frac{t-t_s}{t_o-t_s}=\frac{t_s+\frac{B}{A}}{t_o-t_s}\left[\frac{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}w\right]}{\left\{J_o\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]-\frac{k}{sh}\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}J_1\left[\left(\frac{As^2}{\alpha}\right)^{\frac{1}{2}}\right]\right\}}-1\right]+$$

$$\frac{2}{t_o-t_s}\sum_{n=1}^{\infty}\frac{hs}{k\left[\lambda_n^2+\left(\frac{hs}{k}\right)^2\right]J_o(\lambda_n)}\Bigg[\left(t_o+\frac{B}{A}\right)-$$

$$\frac{\left(t_s+\frac{B}{A}\right)\lambda_n^2}{\left(\lambda_n^2-\frac{As^2}{\alpha}\right)}\Bigg]J_o(\lambda_n w)\,e^{\frac{x}{v}\left[A-\alpha\left(\frac{\lambda_n}{s}\right)^2\right]} \tag{21}$$

Equations (17) and (21) are presented in Figure 2, which plots the temperature at the axis of the tube as a function of the dimensionless distance downstream, the other variables of the problem being treated as dimensionless parameters.

An important aspect of solutions (17) and (21) is that the downstream temperature can increase out of bounds if $A>5.76\,\alpha/s^2$ [isothermal wall, Equation (17)] or if $A>\alpha(\lambda_1/s)^2$ [Equation (21), λ_1 is the smallest root of (8)]. The assumptions of uniform velocity and of heat transport in the radial direction by conduction only are not practical; however, the form of the results developed above should be useful for cases of turbulent flow if a turbulent eddy diffusivity for heat is used in place of the ordinary thermal diffusivity of the fluid.

NOTATION

A, B defined by $q/\rho c = At+B$

J_0, J_1, J_2 = Bessel functions of the first kind and zero, first and second order

N_n = coefficient of various Fourier-Bessel and Dini expansions

V_x, V_y, V_z; V = local velocity in x, y, and z directions; uniform velocity in x direction

c = heat capacity

h = heat transfer coefficient in region exterior to fluid

k = thermal conductivity of fluid

q = rate of heat generation per unit volume

r = radial position coordinate

s = radius of tube

t, t_0, t_s = temperature, temperature at $x=0$, temperature of surroundings ($r>s$)

$w = r/s$

α = thermal diffusivity $= k/\rho c$

β_n = a zero of $J_0(\beta)$

λ_n = a zero of $\lambda J_1(\lambda) - hs/k\, J_0(\lambda)$

ρ = density

τ = time

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